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THE INVERSE OF A TRIDIAGONAL MATRIX

Palmer R. Schlegel, et al

Ballistic Research Laboratories Aberdeen Proving Ground, Maryland

September 1972

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by

Palmer R. Schlegel William Clare Taylor



September 1972

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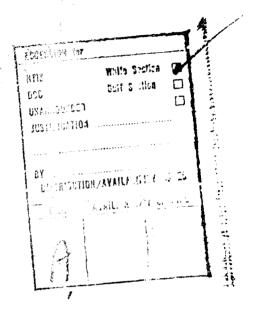
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Security classification of the	NT CONTROL DATA - R & D .
ORIGINATING ACTIVITY (Corporate author)	nd indexing annotation must be entered when the overall report is classified)
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Ballistic Research Laboratories	Unclassified
	21005 N/A
Aberdeen Proving Ground, Maryland	21005 N/A
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Palmer R. Schlegel, William Clare T	laylor
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	74. TOTAL NO. OF PAGES 75. NO. OF REFS
September 1972	
- CONTRACT OR GRANT NO.	M. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO. 1T062110A027	Report No. 1612
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	90. OTHER REPORT NO(S) (Any other numbers that may be assign this report)
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BALLISTIC RESEARCH LABORATORIES

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THE INVERSE OF A TRIDIAGONAL MATRIX

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RDT&E Project No. 1T062110A027

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* REPORT NO. 1612

PRSchlegel/WCTaylor/ats Aberdeen Proving Ground, Md. September 1972

THE INVERSE OF A TRIDIAGONAL MATRIX

ABSTRACT

The closed form inverse of a fairly general tridiagonal matrix is given. The restriction is that the off-diagonal elements in the tridiagonal band be nonzero. If the elements of the matrix are integers, where the upper off-diagonal elements are equal and the lower off-diagonal elements are equal, then an integer multiple of each element of the inverse can be generated by integer arithmetic.

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The closed form inverse of a fairly general tridiagonal matrix is given. The restriction is that the off-diagonal elements in the tridiagonal band be nonzero. If the elements of the matrix are integers, where the upper off-diagonal elements are equal and the lower off-diagonal elements are equal, then an integer multiple of each element of the inverse can be generated by integer arithmetic.

In an earlier note an identity was obtained. Here we present a simpler proof of a more general relation. Multiplication of a vector by a tridiagonal matrix consists of applying a three term recurrence operation to the sequence of elements of the vector. We give the explicit expression for the inverse of the matrix in terms of two vectors, U and V, formed by recursive solution of the homogeneous relation with the zero boundary condition at the beginning and at the end respectively.

FORMS OF MATRIX AND ITS INVERSE

Let B = (b_{ij}) be a tridiagonal matrix such that $b_{i,i+1} \neq 0$ and $b_{i+1,i} \neq 0$, $i = 1, \ldots, n-1$. Define $u_0 = 0$, $u_1 = 1$ and $u_{i+1} = -b_{i,i+1}^{-1}(b_{i,i-1}u_{i-1} + b_{ii}u_i)$ for $1 \leq i \leq n-1$ and $v_{n+1} = 0$, $v_n = 1$ and $v_{j-1} = -b_{j,j-1}^{-1}(b_{jj}v_j + b_{j,j+1}v_{j+1})$ for n+1 > j > 1. Let $\lambda_j^{-1} = b_{j,j-1}v_ju_{j-1} + b_{jj}v_ju_j + b_{j,j+1}v_{j+1}u_j$,

$$c_{ij} = \lambda_j v_j u_i$$
, $i \le j$
= $\lambda_j v_i u_j$, $i > j$,

where

Superscript numerals refer to references found on page 10.

is the inverse of B. Conversely, if B has an inverse, it is obtained by this construction. To see this let $U^T = (u_1, \ldots, u_n)$ and $V^T = (v_1, \ldots, v_n)$, where u_i and v_i are defined above. Then $(BU)^T = (0, \ldots, 0, \xi)$ and $(BV)^T = (\eta, 0, \ldots, 0)$, where ξ and η are some numbers. Let C_j be the jth column of B^{-1} . Since the first j-1 elements of B^{-1} are zero, for some r_j

$$c_{ij} = r_{j}u_{i}$$
, $i \le j$.

Similarly, since the last n-j-1 elements of BC_j are zero, for some s_j

$$c_{ij} = s_j v_i$$
, $i \ge j$.

Thus,

$$r_j^u = c_{jj} = s_j^v$$

or for some λ_i

$$r_i = \lambda_i v_i$$

and

$$s_j = \lambda_j u_j$$
.

The jth element of BC_j is

$$1 = b_{j,j-1}^{c} c_{j-1,j} + b_{jj}^{c} c_{jj} + b_{j,j+1}^{c} c_{j+1,j}$$

$$= b_{j,j-1}^{r} c_{j}^{u} c_{j-1} + b_{j,j}^{r} c_{j}^{u} c_{j} + b_{j,j+1}^{s} c_{j+1}^{s} c_{j+1}^{s}$$

$$= \lambda_{j}^{s} (b_{j,j-1}^{s} c_{j}^{u} c_{j-1}^{s} c_{j}^{s} c_{$$

and this determines λ_{i} .

III. INTEGER ELEMENTS

Suppose the elements of B are integers. Furthermore, suppose

$$b_{i,i+1} = c$$
 and $b_{i+1,i} = d$ for $i = 1,...,n-1$. Let

$$x_{i+1} = -(cdx_{i-1} + b_{ii}x_i),$$

$$y_{j-i} = -(b_{jj}y_j + cdy_{j+1})$$

and

$$w_{j} = cdx_{j-1}y_{j} + b_{jj}x_{j}y_{j} + cdx_{j}y_{j+1},$$

where $x_0 = 0$, $x_1 = 1$, $y_{n+1} = 0$ and $y_n = 1$. It can be shown that

$$w_j^{c}_{ij} = c^{j-i}x_i^{y}_j, i \le j$$

= $d^{i-j}x_j^{y}_i, i > j,$

that is, an integer multiple of each element of the inverse can be generated by integer arithmetic.

REFERENCES

1. P. Schlegel, "The Explicit Inverse of a Tridiagonal Matrix," Math of Comp., Vol 24, 1970, p665.